Non-deterministic Logical Matrices and Their Applications

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General Framework for Finite-valued Logics

- $\mathcal{L}$ — propositional language
- O_k $(k \geqslant 0)$ – the set of k-ary connectives of $\mathcal{L}$
- $\mathcal{W}$ — the set of well-formed formulas of $\mathcal{L}$
- p, q, r — propositional variables
- $\varphi, \psi, \phi, \tau$ — arbitrary formulas
- Γ, Δ — finite sets of formulas of $\mathcal{L}$.

Multi-Valued Semantics

Definition

Let $\mathcal{L}$ be a propositional logic. A **multi-valued semantics** for $\mathcal{L}$ is a triple $\mathcal{S} = (\mathcal{T}, \mathcal{D}, \mathcal{V})$, where:

- $\mathcal{T}$ is a set of **truth values**;
  - $\mathcal{D} \subset \mathcal{T}$ is the set of **designated values**, and
  - $\mathcal{N} = \mathcal{T} - \mathcal{D}$ is the set of **non-designated values**;
- $\mathcal{V}$ is a set of functions $v : \mathcal{W} :\to \mathcal{T}$, called "admissible", or "legal", valuations.

Satisfaction and Validity

Definition

- A formula $\varphi \in \mathcal{W}$ is:
  - **satisfied** by a valuation $v \in \mathcal{V}$, in symbols $v \models \varphi$, if $v(\varphi) \in \mathcal{D}$;
  - **valid**, if it is satisfied by all valuations $v \in \mathcal{V}$.
- A set of formulas $S \subseteq \mathcal{W}$ is:
  - **satisfied** by a valuation $v \in \mathcal{V}$, in symbols $v \models S$, if v satisfies all formulas in S;
  - **valid**, if it is satisfied by all valuations $v \in \mathcal{V}$.
- An (ordinary) **sequent** $\Sigma = \Gamma \Rightarrow \Delta$ is:
  - **satisfied** by a valuation $v \in \mathcal{V}$, in symbols $v \models \Sigma$, iff either $v \not\models \varphi$ for some $\varphi \in \Gamma$ or $v \models \psi$ for some $\psi \in \Delta$;
  - **valid**, if it is satisfied by all valuations $v \in \mathcal{V}$.

Ordinary Logical Matrices

Definition

An (ordinary) **logical matrix** for $\mathcal{L}$ is a triple $\mathcal{M} = (\mathcal{T}, \mathcal{D}, \mathcal{O})$, where

- $\mathcal{T}$ is a non-empty set of truth values;
- $\mathcal{D} \subset \mathcal{T}$ is the set of designated values
- for every k-ary connective $\diamond \in O_k$,
  $\mathcal{O}$ includes its interpretation $\widetilde{\diamond} : \mathcal{T}^k \to \mathcal{T}$.

Example

The matrix for Kleene's 3-valued logic is $\mathcal{M}_K = (\mathcal{T}, \mathcal{D}, \mathcal{O})$, where

$$\mathcal{T} = \{\mathbf{f}, \mathbf{e}, \mathbf{t}\}, \mathcal{D} = \{\mathbf{t}\}, \mathcal{O} = \{\widetilde{\neg}, \widetilde{\vee}, \widetilde{\wedge}\}$$

and the interpretations of connectives are given by:

$\widetilde{\neg}$	**f**	**e**	**t**
	t	**e**	**f**

$\widetilde{\vee}$	**f**	**e**	**t**
f	**f**	**e**	**t**
e	**e**	**e**	**t**
t	**t**	**t**	**t**

$\widetilde{\wedge}$	**f**	**e**	**t**
f	**f**	**f**	**f**
e	**f**	**e**	**e**
t	**f**	**e**	**t**

Valuation in an Ordinary Matrix

A valuation in an ordinary matrix $\mathcal{M}$ is defined **compositionally**:

Definition

A **valuation** in a matrix $\mathcal{M} = (\mathcal{T}, \mathcal{D}, \mathcal{O})$ is a function $v : \mathcal{W} \to \mathcal{T}$ such that:

$$v(\diamond(\psi_1, \dots, \psi_n)) = \widetilde{\diamond}(v(\psi_1), \dots, v(\psi_k))$$

for each k-ary connective $\diamond \in O_k$ and for all $\psi_1, \dots, \psi_k \in \mathcal{W}$.

As a result, any mapping of propositional variables to truth values can be uniquely extended to a legal valuation of all formulas in $\mathcal{M}$.

Nondeterministic Logical Matrices

Definition

A *non-deterministic matrix* (*Nmatrix*) for $\mathcal{L}$ is a triple $\mathcal{M} = (\mathcal{T}, \mathcal{D}, \mathcal{O})$, where:

- $\mathcal{T}$ is a non-empty set of truth values;
- $\mathcal{D} \subset \mathcal{T}$ is the set of designated values;
- for every k-ary connective $\diamond \in O_k$, $\mathcal{O}$ includes its **non-deterministic** interpretation $\widetilde{\diamond} : \mathcal{T}^k \to 2^{\mathcal{T}} - \{\emptyset\}$.

A **legal valuation** in an Nmatrix $\mathcal{M}$ is a function $v : \mathcal{W} \to \mathcal{T}$ such that

$$v(\diamond(\psi_1, \ldots, \psi_k)) \in \widetilde{\diamond}(v(\psi_1), \ldots, v(\psi_k))$$

for each k-ary connective $\diamond \in O_k$ and for all $\psi_1, \ldots, \psi_k \in \mathcal{W}$.

Non-compositionality of a Valuation in an Nmatrix

- A valuation in an Nmatrix is in general **not compositional**, i.e. $v(\diamond(\psi_1, \ldots, \psi_k))$ **is not uniquely determined by** $v(\psi_1), \ldots, v(\psi_k)$
- The value of $v(\diamond(\psi_1, \ldots, \psi_k))$ is selected out of the whole set $\widetilde{\diamond}(v(\psi_1), \ldots, v(\psi_k))$ of the allowed values separately and independently for each instance of the tuple $\langle v(\psi_1), \ldots, v(\psi_k)\rangle$ **(computation level choice)**.

Example

McCarthy's 3-valued matrix differs from Kleene's matrix only in having $\widetilde{\vee}(\mathbf{e},\mathbf{t}) = \widetilde{\wedge}(\mathbf{e},\mathbf{f}) = \mathbf{e}$. Hence their combination can be represented as the Nmatrix $\mathcal{M}_{MK}$ with the truth tables of the form:

$\widetilde{\neg}$	**f**	**e**	**t**
	t	**e**	**f**

$\widetilde{\vee}$	**f**	**e**	**t**
f	**f**	**e**	**t**
e	**e**	**e**	$\{\mathbf{e},\mathbf{t}\}$
t	**t**	**t**	**t**

$\widetilde{\wedge}$	**f**	**e**	**t**
f	**f**	**f**	**f**
e	$\{\mathbf{f},\mathbf{e}\}$	**e**	**e**
t	**f**	**e**	**t**

Then for a legal valuation v we can have

$$v(\alpha_1) = v(\alpha_2) = \mathbf{e},\ \ v(\beta_1) = v(\beta_2) = \mathbf{t}, v(\alpha_1 \vee \beta_1) = \mathbf{e},\ \ v(\alpha_2 \vee \beta_2) = \mathbf{t}$$

Satisfaction, Validity and Consequence Relation

The notions of satisfaction and validity for an Nmatrix $\mathcal{M} = (\mathcal{T}, \mathcal{D}, \mathcal{O})$ coincide with those given before for the multi-valued semantics $\mathcal{S} = (\mathcal{T}, \mathcal{D}, \mathcal{V})$, where $\mathcal{V}$ is the set of all legal valuations in $\mathcal{M}$.

Definition

The consequence relation for $\mathcal{L}$ defined by an Nmatrix $\mathcal{M}$ is the relation $\vdash_{\mathcal{M}}$ between sets of formulas and single formulas in $\mathcal{W}$ such that, for any $T \subseteq \mathcal{W}, \varphi \in \mathcal{W}$

$$T \vdash_{\mathcal{M}} \varphi$$

iff for any valuation v legal in $\mathcal{M}$, $v \models T$ implies $v \models \varphi$.

Nmatrices Versus Ordinary Matrices

- Nmatrices are a generalization of ordinary matrices: the latter correspond to Nmatrices with singletons in all cells, i.e. such that the interpretation $\widetilde{\diamond}$ is singleton-valued for any connective $\diamond$.
- Nmatrices allow us to provide finite semantics for some logics which do not have finite semantics based on ordinary matrices, e.g.:
  - all logics obtained from the classical logic by deleting some rule(s) from its standard sequent calculus,
  - source-processor logics extending the well-known Belnap's approach to collecting and integrating information from many sources,
  - the majority of so-called logics of formal inconsistency (LFI)

Nmatrices Versus Ordinary Matrices

Finite Nmatrices preserve the basic advantages of finite ordinary, deterministic matrices, like:

- decidability
- compactness
- analyticity (effectiveness) (the possibility to extend any partial valuation defined on a subset of $\mathcal{W}$ closed under subformulas to a full valuation)
- existence of a uniform method for developing a corresponding cut-free *n*-sequent calculus for each of them.

Assume $\mathcal{S} = (\mathcal{T}, \mathcal{D}, \mathcal{V})$ is a multi-valued semantics for a propositional language $\mathcal{L}$, with

$$\mathcal{T} = \{t_0, t_1, \ldots, t_{n-1}\} \qquad \mathcal{D} = \{t_k, \ldots, t_{n-1}\}$$

where $1 \leqslant k \leqslant n-1$.

Definition

By an *n*-**sequent** over $\mathcal{L}$ and $\mathcal{T}$ we mean an expression

$$\Sigma = \Gamma_0 \mid \Gamma_1 \mid \ldots \mid \Gamma_{n-1}$$

where, for each i, Γ_i is a finite set of formulas of $\mathcal{L}$.

The bar between the positions corresponding to the values in $\mathcal{N}$ and in $\mathcal{D}$ is often replaced with $\Rightarrow$.

Satisfaction and Validity of n-sequents

Definition

- A valuation $v \in \mathcal{V}$ satisfies the n-sequent $\Sigma = \Gamma_0 \mid \Gamma_1 \mid \ldots \mid \Gamma_{n-1}$, written $v \models \Sigma$, if there is an $i, 0 \leqslant i \leqslant n-1$, such that

$$v(\varphi_i) = t_i \text{ for some } \varphi_i \in \Gamma_i$$

- An n-sequent Σ is valid in $\mathcal{V}$, in symbols $\models_{\mathcal{V}} \Sigma$, if $v \models \Sigma$ for every $v \in \mathcal{V}$.

Fact

A valuation $v \in \mathcal{V}$ satisfies an ordinary sequent $\Gamma \Rightarrow \Delta$ iff v satisfies the n-sequent

$$\Gamma \mid \Gamma \mid \cdots \mid \Gamma \Rightarrow \Delta \mid \Delta \mid \ldots \mid \Delta$$

Alternate Notation — Signed Formulas

Definition

- A *signed formula* over $\mathcal{L}$ and $\mathcal{T}$ is an expression of the form $a : \psi$, where $a \in \mathcal{T}, \psi \in \mathcal{W}$.
- A valuation v in $\mathcal{V}$ *satisfies* a signed formula $a : \psi$, in symbols $v \models a : \psi$, if $v(\psi) = a$.

Signed formulas will be denoted by $\alpha, \beta, \ldots$, and sets of signed formulas — by Ω, Σ, Φ.

Sets of signed formulas are interpreted *disjunctively*:

- A valuation $v \in \mathcal{V}$ satisfies a set of signed formulas Ω, in symbols $v \models \Omega$, iff it satisfies *some* signed formula $\alpha \in \Omega$.
- A set of signed formulas Ω is valid, in symbols $\models \Omega$, iff $v \models \Omega$ for every valuation $v \in \mathcal{V}$.

Equivalence of n-sequents and Sets of Signed Formulae

Fact

A sequent Σ of the form

$$\Gamma_0 \mid \Gamma_1 \mid \ldots \mid \Gamma_{n-1}$$

is satisfied by a valuation v (valid) if and only if the set

$$\Omega = \{t_0 : \Gamma_0, t_1 : \Gamma_1, \ldots, t_{n-1} : \Gamma_{n-1}\}$$

of signed formulas is satisfied by a valuation v (valid).

n-sequent Calculus for Logic Based on Finite-Valued Nmatrix

Theorem

Let $\mathcal{M} = (\mathcal{T}, \mathcal{D}, \mathcal{O})$ be an n-valued Nmatrix. Then the n-sequent system defined below is sound and complete for the consequence relation $\vdash_{\mathcal{M}}$.

- **Axioms**: $\{a : \varphi \mid a \in \mathcal{T}\}$ *for any formula* $\varphi \in \mathcal{W}$.
- **Structural inference rules:** *Weakening.*
- **Logical inference rules:**

$$\frac{\Omega, a_1 : \varphi_1 \quad \ldots \quad \Omega, a_m : \varphi_m}{\Omega, b_1 : \diamond(\varphi_1, \ldots, \varphi_m), \ldots, b_k : \diamond(\varphi_1, \ldots, \varphi_m)}$$

for any m-ary connective $\diamond \in \mathcal{O}$*, and any* $a_1, a_2, \ldots, a_m$, $b_1, b_2, \ldots, b_k \in \mathcal{T}$ *such that*

$$\widetilde{\diamond}(a_1, \ldots, a_m) = \{b_1, \ldots, b_k\}$$

Exemplary Rule for Disjunction in $\mathcal{M}_{MK}$

Example

In the Nmatrix $\mathcal{M}_{MK}$, we have

$$\widetilde{\vee}(\mathbf{e}, \mathbf{t}) = \{\mathbf{e}, \mathbf{t}\}$$

This gives rise to the rule

$$\frac{\Omega, \mathbf{e} : \varphi_1 \quad \Omega, \mathbf{t} : \varphi_2}{\Omega, \mathbf{e} : \varphi_1 \vee \varphi_2, \mathbf{t} : \varphi_1 \vee \varphi_2}$$

or, in the sequent notation

$$\frac{\Gamma_{\mathbf{f}} \mid \Gamma_{\mathbf{e}}, \varphi_1 \Rightarrow \Gamma_{\mathbf{t}} \qquad \Gamma_{\mathbf{f}} \mid \Gamma_{\mathbf{e}} \Rightarrow \Gamma_{\mathbf{t}}, \varphi_2}{\Gamma_{\mathbf{f}} \mid \Gamma_{\mathbf{e}}, \varphi_1 \vee \varphi_2 \Rightarrow \Gamma_{\mathbf{t}}, \varphi_1 \vee \varphi_2}$$

Sufficiently Expressive Languages

Definition

Let $\mathcal{T} = \{a_0, a_1, \ldots, a_{n-1}\}$. The language $\mathcal{L}$ is **sufficiently expressive** for the semantics $\mathcal{S} = (\mathcal{T}, \mathcal{D}, \mathcal{V})$ iff, for any $i = 0, \ldots, n-1$, there exist natural numbers $l_i, m_i \geqslant 0$ and single-variable formulas

$$A^i_j, B^i_k \in \mathcal{W}, j = 1, \ldots, l_i, k = 1 \ldots, m_i$$

such that, for any valuation $v \in \mathcal{V}$ and any formula $\varphi \in \mathcal{W}$,

$$\begin{aligned} v(\varphi) = a_i \quad \Leftrightarrow \quad & v(A^i_1\varphi), \ldots, v(A^i_{l_i}\varphi) \in \mathcal{N} \\ \& \quad & v(B^i_1\varphi), \ldots, v(B^i_{m_i}\varphi) \in \mathcal{D} \end{aligned}$$

Translation of n-sequents to Ordinary Ones

Theorem

Let $\mathcal{L}$ be a language sufficiently expressive for an n-valued semantics $\mathcal{S} = (\mathcal{T}, \mathcal{D}, \mathcal{V})$. Then there is an algorithm that translates any n-sequent Σ over $\mathcal{L}$ to a set $Ord(\Sigma)$ of ordinary sequents over $\mathcal{L}$ such that, for any any valuation $v \in \mathcal{V}$

$$v \models \Sigma \text{ iff } v \models \Sigma' \text{ for every } \Sigma' \in Ord(\Sigma)$$

Translation of an n-sequent Calculus to Ordinary One

Theorem

Let $\mathcal{C}$ be a sound and complete n-sequent calculus for a language $\mathcal{L}$ sufficiently expressive for $\mathcal{S}$, and let $Ord(\mathcal{C})$ be the calculus of ordinary sequents for $\mathcal{L}$ consisting of:

- **Axioms**:

$$\{Ord(A) \mid A \text{ is an axiom of } \mathcal{C}\}$$

- **Inference rules**:

$$\{\frac{Ord(\mathcal{P})}{\Sigma'} \mid \ \Sigma' \in Ord(R) \text{ for some rule } \frac{\mathcal{P}}{R} \text{ in } \mathcal{C}\}$$

Then $Ord(\mathcal{C})$ is a sound and complete ordinary sequent calculus for $\mathcal{L}, \mathcal{S}$.

Development of Ordinary Sequent Calculi for Multi-valued Logics Based on Nmatrices

1. Out of an n-valued semantics for a propositional logic based on an Nmatrix, we develop a sound and complete n-sequent calculus, given by a uniform, generic template.
2. For logics with n-valued semantics that have sufficiently expressive languages, each n-sequent can be translated to an equivalent set of ordinary sequents.
3. Based on the above translation, a complete n-sequent calculus for a logic with a sufficiently expressive language can be translated to a sound and complete calculus of ordinary sequents for the considered n-valued logic.
4. If the language of the logic is not sufficiently expressive, it can be made sufficiently expressive by adding the appropriate construct ensuring its expressiveness, e.g. multiple signs or unary J_k operators of Rosser-Turquette logic

Paradigmatic example: Rosser-Turquette Logic

Example

The n-valued Rosser-Turquette logic with the "truth threshold" s is defined by the ordinary matrix $\mathcal{M}_{RT} = (\mathcal{T}, \mathcal{D}, \mathcal{O})$, where:

$$\mathcal{T} = \{0, 1, \ldots, n{-}1\},\ \ \mathcal{D} = \{s, s{+}1, \ldots, n{-}1\}, \mathcal{O} = \ \{\neg, \vee, \wedge, J_0, J_1, \ldots, J_{n-1}\}$$

where $\widetilde{J_k} t = n - 1$ if $t = k$, 0 otherwise.The language is sufficiently expressive, because

$$v(\varphi) = i \Leftrightarrow v(J_i \varphi) \in \mathcal{D}$$

for any valuation v, formula φ and $0 \leqslant i \leqslant n - 1$. Hence any n-sequent translates to a single one-sided ordinary sequent:

$$Ord(\Gamma_0 \mid \ldots \mid \Gamma_{s-1} \Rightarrow \Gamma_s \mid \ldots \mid \Gamma_{n-1}) = \{\Rightarrow J_0\Gamma_0, J_1\Gamma_1, \ldots, J_{n-1}\Gamma_{n-1}\}$$

McCarthy-Kleene Nmatrix

Consider again the McCarthy-Kleene Nmatrix $\mathcal{M}_{MK} = (\mathcal{T}, \mathcal{D}, \mathcal{O}_{MK})$, where

$$\mathcal{T} = \{\mathbf{f}, \mathbf{e}, \mathbf{t}\}, \mathcal{D} = \{\mathbf{t}\}$$

and $\mathcal{O}$ is given by the following truth tables:

$\tilde{\neg}$	**f**	**e**	**t**
	t	**e**	**f**

$\widetilde{\vee}$	**f**	**e**	**t**
f	**f**	**e**	**t**
e	**e**	**e**	$\{\mathbf{e}, \mathbf{t}\}$
t	**t**	**t**	**t**

$\widetilde{\wedge}$	**f**	**e**	**t**
f	**f**	**f**	**f**
e	$\{\mathbf{f}, \mathbf{e}\}$	**e**	**e**
t	**f**	**e**	**t**

Sufficient Expressiveness

The language of $\mathcal{M}_{MK}$ is sufficiently expressive for its semantics, because:

$$\begin{array}{lll} v(\varphi) = \mathbf{t} & \text{iff} & v(\varphi) \in \mathcal{D} \\ v(\varphi) = \mathbf{e} & \text{iff} & v(\varphi) \in \mathcal{N} \;\&\; v(\neg\varphi) \in \mathcal{N} \\ v(\varphi) = \mathbf{f} & \text{iff} & v(\varphi) \in \mathcal{N} \;\&\; v(\neg\varphi) \in \mathcal{D} \end{array}$$

The set $Ord(\Sigma)$ of ordinary sequents equivalent to a 3-valued sequent $\Sigma = \Gamma_{\mathbf{f}} \mid \Gamma_{\mathbf{e}} \Rightarrow \Gamma_{\mathbf{t}}$ is the set of all ordinary sequents of the form

$$\Gamma'_{\mathbf{f}}, \Gamma'_{\mathbf{e}}, \neg\Gamma''_{\mathbf{e}} \Rightarrow \Gamma_{\mathbf{t}}, \neg\Gamma''_{\mathbf{f}}$$

where $\Gamma'_i \uplus \Gamma''_i = \Gamma_i$ for $i = \mathbf{e}, \mathbf{f}$.

Logics for Collecting and Processing Information from Multiple Sources

L_C — classical logic; $\mathcal{A}$ — atoms of L_C; $\mathcal{F}$ — formulas of L_C.

A **source-processor structure** consists of:

- A set S of **sources** providing information about formulas of L_C. For any $\varphi \in \mathcal{F}$, a source $s \in S$ can:
  - assign 1 to φ to say that formula φ is true;
  - assign 0 to φ to say that formula φ is false;
  - assign nothing to φ if it knows nothing about its value
- A **processor** P, which:
  - collects information from the sources,
  - combines it according to some strategy,
  - derives the resulting valuation of formulas in $\mathcal{F}$

Possible Combinations of Information

For any formula $\varphi \in \mathcal{F}$, processor P can obtain four possible combinations of information from the sources:

- P obtains the information that φ is true, but no information that φ is false
- P obtains the information that φ is false, but no information that φ is true
- P obtains both the information that φ is true and the information that φ is false
- P obtains no information on φ at all

Logical domain for the processor: four logical values corresponding to the four possible combinations of information from the sources:

$$\mathbf{t} = \{1\},\ \mathbf{f} = \{0\},\ \top = \{0,1\},\ \bot = \emptyset,$$

where:

- 1 and 0 represent the classical truth values "true" and "false", respectively,
- $\top$ represents inconsistent information,
- $\bot$ represents absence of information

Designated values: $\mathbf{t}$ and $\top$

The source-processor approach was initiated by Belnap, who considered the case when the sources provide information about atomic formulas only. The resulting valuations defined by the processor observe Dunn-Belnap 4-valued matrix $\mathcal{M}_B^4 = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where

$$\mathcal{V} = \{\mathbf{f}, \bot, \top, \mathbf{t}\}, \mathcal{D} = \{\top, \mathbf{t}\}$$

and the interpretations of the connectives are given by:

$\tilde{\neg}$	
$\mathbf{f}$	$\mathbf{t}$
$\bot$	$\bot$
$\top$	$\top$
$\mathbf{t}$	$\mathbf{f}$

$\tilde{\vee}$	$\mathbf{f}$	$\bot$	$\top$	$\mathbf{t}$
$\mathbf{f}$	$\mathbf{f}$	$\mathbf{t}$	$\top$	$\mathbf{t}$
$\bot$	$\bot$	$\bot$	$\mathbf{t}$	$\mathbf{t}$
$\top$	$\top$	$\mathbf{t}$	$\top$	$\mathbf{t}$
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$

$\tilde{\wedge}$	$\mathbf{f}$	$\bot$	$\top$	$\mathbf{t}$
$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$
$\bot$	$\mathbf{f}$	$\bot$	$\mathbf{f}$	$\bot$
$\top$	$\mathbf{f}$	$\mathbf{f}$	$\top$	$\top$
$\mathbf{t}$	$\mathbf{f}$	$\bot$	$\top$	$\mathbf{t}$

Extending Belnap's Model

Belnap's model can be extended by allowing the sources to provide information about complex formulas, too. This leads to possibly non-deterministic results of combining information about atomic formulas with information about complex formulas.

Example

If one source says that α, β are false, then the processor can conclude that $\alpha \vee \beta$ is false. However, another source can simply say that $\alpha \vee \beta$ is true — and then we only know that $\alpha \vee \beta$ is either true or false, but do not know which of the above values it takes.

It turns out that the resulting non-determinism cannot be described with ordinary logical matrices, and can only be handled using Nmatrices.

Method of Generating a Processor Valuation

Denote:
$v_s : \mathcal{F} \rightsquigarrow \{0,1\}$ — partial valuation of formulas defined by a source $s \in S$
$v_P : \mathcal{F} \to \mathcal{P}(\{0,1\})$ — valuation of formulas defined by the processor P

- The processor collects information from the sources and:
  - puts 0 in $v_P(\varphi)$ iff there is a source s such that $v_s(\varphi) = 0$;
  - puts 1 n $v_P(\varphi)$ iff there is a source s such that $v_s(\varphi) = 1$;
- After collecting the information as above, the processor extends the collected information with conclusions drawn from it using the truth tables of classical logic applied forward and backward.

Example

- If after collecting the information from the sources the processor P obtains a valuation v_P with $1 \in v_P(\alpha)$, then from the truth table of classical disjunction applied forward P can conclude it should put 1 in $v_P(\alpha \vee \beta)$ for any formula β.
- On the other hand, if $1 \in v_P(\gamma \wedge \delta)$, then from the truth table of classical conjunction applied backward the processor can conclude that it should put 1 in both $v_P(\gamma)$ and $v_P(\delta)$.

Nondeterminism of the Processor Valuation

Example

Suppose $v_s(\alpha_i)$ is either 0 or undefined for $i = 1, 2$ and each source $s \in S$. Then $0 \in v_P(\alpha_i)$ for $i = 1, 2$, and so $0 \in v_P(\alpha_1 \vee \alpha_2)$ by the truth table of disjunction.

However, if there is a source s_0 such that $v_{s_0}(\alpha \vee \beta) = 1$, then $1 \in v_P(\alpha_1 \vee \alpha_2)$ too. As a result, $v_P(\alpha \vee \beta) = \{0, 1\}$ if such an s_0 exists, and $\{0\}$ otherwise. Thus in general we can only say that

$$v_P(\alpha_1 \vee \alpha_2) \in \{\mathbf{f}, \top\}$$

— which means that we need Nmatrices to describe the considered framework.

Nmatrix Representation of Processor Valuations

Theorem

The set of processor valuations coincides with the set of legal valuations in the four-valued Nmatrix $\mathcal{M}_I^4$ with the truth tables

$\widetilde{\neg}$	
$\mathbf{f}$	$\{\mathbf{t}\}$
$\bot$	$\{\bot\}$
$\top$	$\{\top\}$
$\mathbf{t}$	$\{\mathbf{f}\}$

$\widetilde{\vee}$	$\mathbf{f}$	$\bot$	$\top$	$\mathbf{t}$
$\mathbf{f}$	$\{\mathbf{f},\top\}$	$\{\mathbf{t},\bot\}$	$\{\top\}$	$\{\mathbf{t}\}$
$\bot$	$\{\mathbf{t},\bot\}$	$\{\mathbf{t},\bot\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$
$\top$	$\{\top\}$	$\{\mathbf{t}\}$	$\{\top\}$	$\{\mathbf{t}\}$
$\mathbf{t}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$

$\widetilde{\wedge}$	$\mathbf{f}$	$\bot$	$\top$	$\mathbf{t}$
$\mathbf{f}$	$\{\mathbf{f}\}$	$\{\mathbf{f}\}$	$\{\mathbf{f}\}$	$\{\mathbf{f}\}$
$\bot$	$\{\mathbf{f}\}$	$\{\mathbf{f},\bot\}$	$\{\mathbf{f}\}$	$\{\mathbf{f},\bot\}$
$\top$	$\{\mathbf{f}\}$	$\{\mathbf{f}\}$	$\{\top\}$	$\{\top\}$
$\mathbf{t}$	$\{\mathbf{f}\}$	$\{\mathbf{f},\bot\}$	$\{\top\}$	$\{\mathbf{t},\top\}$

where

$$\mathbf{f} = \{0\}, \bot = \{\emptyset\}, \top = \{0,1\}, \mathbf{t} = \{1\}$$

Non-trivial Reasoning with Inconsistencies

- Within classical logic, inconsistency leads to the trivialization, as everything becomes derivable:

$$A, \neg A \vdash B$$

- **A paraconsistent logic** allows contradictory but non-trivial theories:

Definition

A propositional logic $\mathbf{L} = \langle \mathcal{L}, \vdash \rangle$ is **paraconsistent** if there are $\mathcal{L}$-formulas A, B such that

$$A, \neg A \nvdash B$$

- Divide propositions into two sorts: consistent and inconsistent.
- Reflect this classification within the language. In **C-systems**, this is done by using a special connective $\circ$ with the intuitive meaning of $\circ A$ being **"A is consistent"**.
- Restrict the explosive character of contradictions: they should be explosive only for **consistent** formulas:

$$\mathbf{A}, \neg\mathbf{A} \vdash \mathbf{B} \quad \Rightarrow \quad \mathbf{A}, \neg\mathbf{A}, \circ\mathbf{A} \vdash \mathbf{B}$$

The system **BK** extends the positive fragment of classical logic with the following axioms:

(**t**) $\neg\mathbf{A} \vee \mathbf{A}$

(**b**) $\circ\mathbf{A} \supset ((\mathbf{A} \wedge \neg\mathbf{A}) \supset \mathbf{B})$

(**k**) $\circ\mathbf{A} \vee (\mathbf{A} \wedge \neg\mathbf{A})$

Other Axioms of C-systems

The axioms used commonly for C-systems include the following set ADC of axioms, where $\sharp \in \{\wedge, \vee, \supset\}$:

$(\mathbf{c})$ $\neg\neg A \supset A$	$(\mathbf{e})$ $A \supset \neg\neg A$
$(\mathbf{i_1})$ $\neg\circ A \supset A$	$(\mathbf{i_2})$ $\neg\circ A \supset \neg A$
$(\mathbf{a}_\sharp)$ $(\circ A \wedge \circ B) \supset \circ(A\sharp B)$	$(\mathbf{a}_\neg)$ $\circ A \supset \circ\neg A$
$(\mathbf{o}^1_\sharp)$ $\circ A \supset \circ(A\sharp B)$	$(\mathbf{o}^2_\sharp)$ $\circ B \supset \circ(A\sharp B)$

For $S \subseteq ADC$, by BK[S] we denote the system obtained from BK by adding the axioms in S.

- Truth-values: $\langle x, y \rangle \in \{0,1\}^2$, where $x = 1$ iff A is "true", and $y = 1$ iff $\neg A$ is "true":

$$t = \langle 1, 0 \rangle, f = \langle 0, 1 \rangle, \top = \langle 1, 1 \rangle, \bot = \langle 0, 0 \rangle$$

  Designated values: $\{\langle 1, \_ \rangle\} = \{t, \top\}$

- $\neg \langle x, y \rangle = \{\langle y, \_ \rangle\}$
- $\langle x_1, y_1 \rangle \sharp \langle x_2, y_2 \rangle = \{\langle x_1 \sharp x_2, \_ \rangle\}$ for $\sharp \in \{\wedge, \vee, \supset\}$, where _ runs over $\{0, 1\}$.

- Axiom (**t**) $\mathbf{(}A \vee \neg A\mathbf{)}$ rules out the truth-value $\bot$.
- Axiom (**b**) $\mathbf{(}\circ A \supset ((A \wedge \neg A) \supset B)\mathbf{)}$ implies: $\circ\top = \{f\}$.
- Axiom (**k**) $\mathbf{(}\circ A \vee (A \wedge \neg A)\mathbf{)}$ implies $\circ t = \circ f = \{t, \top\}$.

The Nmatrix $\mathcal{M}^3_{\mathbf{BK}}$

The Nmatrix $\mathcal{M}^3_{\mathbf{BK}}$ is obtained by imposing the above semantic effects of the axioms **(t), (b), (k)** on the nondeterministic truth tables of $\neg, \vee, \wedge, \supset$ corresponding to their interpretation defined above:

a	$\neg a$	$\circ a$
t	$\{f\}$	$\{t, \top\}$
$\top$	$\{t, \top\}$	$\{f\}$
f	$\{t, \top\}$	$\{t, \top\}$

$\wedge$	t	$\top$	f
t	$\{t, \top\}$	$\{t, \top\}$	$\{f\}$
$\top$	$\{t, \top\}$	$\{t, \top\}$	$\{f\}$
f	$\{f\}$	$\{f\}$	$\{f\}$

$\vee$	t	$\top$	f
t	$\{t, \top\}$	$\{t, \top\}$	$\{t, \top\}$
$\top$	$\{t, \top\}$	$\{t, \top\}$	$\{t, \top\}$
f	$\{t, \top\}$	$\{t, \top\}$	$\{f\}$

$\supset$	t	$\top$	f
t	$\{t, \top\}$	$\{t, \top\}$	$\{f\}$
$\top$	$\{t, \top\}$	$\{t, \top\}$	$\{f\}$
f	$\{t, \top\}$	$\{t, \top\}$	$\{t, \top\}$

Adding Axioms in *ADC* to BK

Example

The axiom $(o^1_\vee)$ $\circ A \supset \circ(A \vee B)$ corresponds to the following semantic conditions:

$$t \vee t = t \vee f = t \vee \top = \{t\} \qquad f \vee t = f \vee \top = \{t\}$$

After imposing them on $\mathcal{M}^3_{\mathbf{BK}}$, we obtain its refinement — the Nmatrix $\mathcal{M}^3[o^1_\vee]$:

$\vee$	t	$\top$	f
t	$\{t,\top\}$	$\{t,\top\}$	$\{t,\top\}$
$\top$	$\{t,\top\}$	$\{t,\top\}$	$\{t,\top\}$
f	$\{t,\top\}$	$\{t,\top\}$	$\{f\}$

$\Rightarrow$

$\vee$	t	$\top$	f
t	$\{\mathbf{t}\}$	$\{t\}$	$\{t\}$
$\top$	$\{t,\top\}$	$\{t,\top\}$	$\{t,\top\}$
f	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$	$\{f\}$

Theorem

For coherent $S \subseteq ADC$, let $\mathcal{M}^3_{\mathbf{BK}}[S]$ be the simplest refinement of $\mathcal{M}^3_{\mathbf{BK}}$ that satisfies all the semantic conditions induced by the axioms in S. Then, for any theory T and any formula A

$$T \vdash_{\mathcal{M}^3_{\mathbf{BK}}[S]} A \text{ iff } T \vdash_{\mathbf{BK}[S]} A$$

Corresponding Cut-free Sequent Calculi

- The Nmatrix $\mathcal{M}^3_{\mathbf{BK}}$ is finite.
- The language we use is sufficiently expressive, because:
  - $v(A) = t$ iff $\neg A \Rightarrow$ is true in v.
  - $v(A) = f$ iff $A \Rightarrow$ is true in v.
  - $v(A) = \top$ iff $\Rightarrow A$ and $\Rightarrow \neg A$ are both true in v.
  - $v(A) \in \{f, \top\}$ iff $\Rightarrow \neg A$ is true in v.
  - $v(A) \in \{t, \top\}$ iff $\Rightarrow A$ is true in v.
  - $v(A) \in \{t, f\}$ iff $A, \neg A \Rightarrow$ is true in v.
- Hence we can use the general method presented before to provide a sound and complete, cut-free ordinary sequent calculus C^3 for $\mathcal{M}^3_{\mathbf{BK}}$.
- For any $S \subseteq ADC$, C^3 can be extended in a modular way to a complete ordinary sequent calculus $C^3[S]$ for $\mathcal{M}^3_{\mathbf{BK}}[S]$ (equivalent to $BK[S]$) by adding sequent rules corresponding to the semantic conditions induced by the axioms in S.

Handling of Axioms (l), (d)

The literature on C-systems considers also the axioms

$$\textbf{(l)}\ \neg(\mathbf{A} \wedge \neg\mathbf{A}) \supset \circ\mathbf{A} \qquad \textbf{(d)}\ \neg(\neg\mathbf{A} \wedge \mathbf{A}) \supset \circ\mathbf{A}$$

which cannot be handled using finite Nmatrices. However, systems containing these axioms can be given semantics based on an infinite Nmatrices, which can be then used to generate for them in a modular way uniform sequent calculi enjoying cut-admissibility .

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