Dependence and Independence of quantifiers

Gabriel Sandu

Dependence in logic

- Goldfarb (1979):

  The connection between quantifiers and choice functions or, more precisely, between quantifier-dependence and choice functions, is the heart of how classical logicians in the twenties viewed the nature of quantification." (Goldfarb 1979, p. 357).

Dependence in independence in logic (Terrence Tao 2007)

- We can express in first-order logic the following statements

1. For every x, there exists a y depending on x such that $B(x,y)$:

$$\forall x \exists y B(x,y)$$

2. For every x, there exists a y independent of x such that $B(x,y)$:

$$\exists y \forall x B(x,y)$$

3. For every x, z, and w there exists a y depending only on x and z such that $Q(x,z,w,y)$:

$$\forall x \forall z \exists y \forall w Q(x,z,w,y)$$

Example: Continuity

- A function f is continuous at a point x_0 if given any $\varepsilon > 0$ one can choose $\delta > 0$ so that for all y, when x_0 is within distance δ from y, then $f(x_0)$ is within distance ε from $f(y)$, i.e.,

$$|x_0 - y| < \delta \to |f(x_0) - f(y)| < \varepsilon$$

- The general form is

$$\forall x_0 \forall \varepsilon \exists \delta \forall y R(x_0, \varepsilon, \delta, y)$$

- The choice of δ depends on both x_0 and ε.

Uniform continuity

- Compare continuity

$$\forall x_0 \forall \varepsilon \exists \delta \forall y R(x_0, \varepsilon, \delta, y)$$

with

- Uniform continuity

$$\forall \varepsilon \exists \delta \forall x_0 \forall y R(x, \varepsilon, \delta, y)$$

where the choice of δ depends only on ε.

Nonfirst-orderisability (Tao)

- But one cannot always express

  For every x and z, there exists a y depending only on x and a w depending only on z such that $Q(x, z, y, w)$ is true

Tao's conclusion:

> It seems to me that first order logic is limited by the linear (and thus totally ordered) nature of its sentences; every new variable that is introduced must be allowed to depend on all the previous variables introduced to the left of that variable. This does not fully capture all of the dependency trees of variables which one deals with in mathematics.

Henkin quantifiers (Henkin 1961)

- For every x and z, there exists a y depending only on x and a w depending only on z such that $Q(x, z, y, w)$ is true

$$\begin{pmatrix} \forall x & \exists y \\ \forall z & \exists w \end{pmatrix} Q(x, z, y, w)$$

Independence-Friendly Logic (IF logic)

- Quantifiers and connectives of the form

$$(\exists x/W),\ \ (\forall x/W),\ \ (\vee/W),\ \ (\wedge/W)$$

with the interpretation: "the choice of x is independent of the values of the variables in W"

- When $W = \emptyset$, we recover the standard quantifiers

- The Henkin formula

$$\begin{pmatrix} \forall x & \exists y \\ \forall z & \exists w \end{pmatrix} Q(x,z,y,w)$$

is expressed by:

$$\forall x \forall z (\exists y/\{z\})(\exists w/\{x,y\}) Q(x,z,y,w).$$

Henkin quantifiers vs. IF logic

- A Henkin quantifier is a “block”

- In an IF formula, an independent quantifier can occur anywhere, e.g.

$$\forall x_0 \forall x_2 (x_0 \neq x_2 \vee (\exists x_1/\{x_0\}) x_0 = x_1)$$

- In IF logic syntactical scope and “dependence” are splitted.

Dependence

- Let Qx/X and Qy/Y be two quantifiers in an IF formula such that Qx/X is in the (syntactical) scope of Qy/Y

- We say that Qx/X depends on Qy/Y if $y \notin X$.

Henkin Prefixes

- A Henkin prefix contains the quantifiers

$$\forall x, \forall y, (\exists u/U), (\exists v/V)$$

such that:

- $(\exists u/U)$ depends on x and not on y

- $(\exists v/V)$ depends on y and not on x and u.

- Thus

$$\forall x \forall y (\exists u/\{y\})(\exists v/\{x, u\})$$

is a Henkin prefix.

Signaling prefixes

- A signaling prefix contains three quantifiers

  $$(\forall u/U), (\exists v/V), (\exists w/W)$$

  such that:

- $(\exists v/V)$ depends on u

- $(\exists w/W)$ depends on v but not on u

- For example

  $$\forall u \exists v (\exists w/\{u\})$$

  is a signaling prefix.

- A signaling prefix breaks knowledge memory.

Game-theoretical semantics (Hintikka)

- Let φ denote

$$\forall x \exists y (\exists z / \{x\}) Q(x,y,z)$$

- Given a model $\mathbb{M}$ which interprets Q, a semantical game of imperfect information $G(\mathbb{M}, \varphi)$.

- The players are Abelard ($\forall$) and Eloise ($\exists$).

- $\forall$ chooses $a \in M$ to be the value of x

- $\exists$ chooses $b \in M$ and then $c \in M$ to be the values of y and z, respectively.

- When choosing c, $\exists$ does not know the value of a.

Winning a play

- $\exists$ wins the play (a, b, c) if $Q(a, b, c)$ holds in $\mathbb{M}$.

- $\forall$ wins the play $(a, b, c,)$ if $Q(a, b, c)$ does not hold in $\mathbb{M}$

Strategies

- Recall the game $G(\mathbb{M}, \varphi)$ with φ

$$\forall x \exists y (\exists z / \{x\}) Q(x, y, z)$$

- A strategy for $\forall$ reduces to an individual a in M.

- A strategy for $\exists$ consists of a pair (f_y, g_z) such that $f_y : M \to M$ and $g_z : M \to M$.

- A strategy (f_y, g_z) is winning for $\exists$ if for all (strategies) a played by $\forall$:

$$Q(a, f_y(a), g_z(f(a)))$$

holds in $\mathbb{M}$.

Truth and falsity

- $M \models^{+}_{GTS} \varphi$ iff there is a wining strategy for $\exists$ in $G(\mathbb{M}, \varphi)$

- $M \models^{-}_{GTS} \varphi$ iff there is a wining strategy for $\forall$ in $G(\mathbb{M}, \varphi)$

Example of signaling prefixes (Hodges)

- We insert one dummy variable in the sentence

$$\forall x(\exists y/\{x\})x = y$$

to obtain

$$\forall x\exists z(\exists y/\{x\})x = y$$

- For any model $\mathbb{M}$ we have:

$$\forall x(\exists y/\{x\})x = y \Leftrightarrow \exists y\forall x(x = y)$$

and

$$\forall x\exists z(\exists y/\{x\})x = y \Leftrightarrow \exists f\exists g\forall x(x = g(f(x)))$$

Example of signaling prefixes

- The sentences φ_{inf}

$$\forall u \exists v (\exists w/\{u\})(u = w \wedge c \neq v)$$

  defines infinity on any structure which interprets c.

- φ_{inf} is equivalent with

$$\exists f \exists g \forall x (x = g(f(x)) \wedge c \neq f(x))$$

Lewis signaling problems

- A Sender (C) and a Receiver (A)
- C observes one of several states s and sends a "signal" t to A, who does not see s.
- After receiving t, A performs one of several alternative actions a (responses).
- Every state s has a corresponding best response $b(s)$
- A word or "signal" gets its meaning in virtue of its role in the solution of various signaling problems.

Lewis signaling problems as cooperative games

- A set S of situations, a set Σ of signals, and a set R of responses

- There are as many signals as states:

$$| S |=| \Sigma |$$

- A function $b : S \rightarrow R$ which maps each situation to its best response.

- An encoding function $f : S \rightarrow \Sigma$ employed by C

- A decoding function $g : \Sigma \rightarrow R$ employed by A.

Signaling systems

- A *signaling system* is a pair (f, g) of encoding and decoding functions such that for every state s : $g(f(s)) = b(s)$.

- When (f, g) is settled, each signal acquires a "meaning": the action which is associated with it.

Definability of signaling systems in IF logic

- The IF sentence φ_{sig}

  $\forall x \exists z (\exists y / \{x\}) \{ (S(x) \to (\Sigma(z) \wedge R(y) \wedge B(x,y)) \}$

- The model $\mathbb{M}$

  $$\mathbb{M} = (M, S^M, \Sigma^M, R^M, B^M)$$

  where

  $$M = \{s_1, ..., s_n, t_1, ..., t_m, a_1, ..., a_n\}$$

  and

  $$\begin{array}{rcl} S^M & = & \{s_1, ..., s_n\} \\ \Sigma^M & = & \{t_1, ..., t_m\} \\ R^M & = & \{a_1, ..., a_n\} \\ B^M & = & \{(s_1, a_1), ..., (s_n, a_n)\} \end{array}$$

- φ_{sig} and $\mathbb{M}$ determine a semantical game $G(\mathbb{M}, \varphi_{sig})$ of imperfect information

Monty Hall

Monty Hall shows C three closed doors: behind one of them there is a prize, the other two are empty. C chooses a door. Monty Hall opens any of the other doors, which is empty. Then she asks C whether he would like to switch the doors, and choose the remaining one which is closed. Is it in C's interest to do it? (Richard Isaac, *The Pleasures of Probability* 1995)

Monty Hall in IF logic

- It is encoded by the game $G(M,\varphi_{MH})$ where φ_{MH} is

  $\forall x(\exists y/\{x\})\forall z[x \neq z \wedge y \neq z \rightarrow (\exists t/\{x\})x = t]$

  and

$$M = \{D_1, D_2, D_3\}$$

Basic result on IF prefixes (Sevenster 2014, Barbero, 2014)

- Every prefix which is neither signaling nor branching is equivalent to an ordinary FOL prefix.

Operations on prefixes: Quantifier swapping

- The prefixes

$$\Pi(Qu/U)(Qv/V \cup u)\Pi'$$

and

$$\Pi(Qv/V - u)(Qu/U \cup v)\Pi'$$

are equivalent.

Operations on prefixes: Dropping slash sets

- The prefixes

$$(\forall v/V)\Pi \quad and \quad \forall v\Pi$$

  are equivalent.

Operations on prefixes: Dropping slash sets

- The sentential prefixes

  $$\Pi(Qv/V)\Pi'$$

  and

  $$\Pi Qv\Pi'$$

  are equivalent whenever V contains variables that are quantified only by existential quantifiers in Π.

An example

- Recall the Monty Hall sentence φ_{MH} :

  $\forall x(\exists y/\{x\})\forall z[x \neq z \wedge y \neq z \rightarrow (\exists t/\{x\})x = t]$

- Notice that it is neither Henkin nor signaling.

- We first pull the existential quantifier out:

  $\forall x(\exists y/\{x\})\forall z(\exists t/\{x\})[x \neq z \wedge y \neq z \rightarrow x = t]$

- We swap the first two quantifiers:

  $\exists y(\forall x/\ \{x)\})\forall z(\exists t/\{x\})[x \neq z \wedge y \neq z \rightarrow x = t]$

- We drop the slash set of the universal quantifier:

$$\exists y \forall x \forall z(\exists t/\{x\})[x \neq z \wedge y \neq z \rightarrow x = t]$$

- We swap the two universal quantifiers:

$$\exists y \forall z \forall x (\exists t/\{x\})[x \neq z \wedge y \neq z \to x = t]$$

- We swap the universal and existential quantifiers:

$$\exists y \forall z \exists t (\forall x/\,\{x\})[x \neq z \wedge y \neq z \to x = t]$$

- Finally we drop the slashed set of the universal quantifier:

$$\exists y \forall z \exists t \forall x [x \neq z \wedge y \neq z \to x = t]$$

Functional dependence in natural language

1. Every country has a tyrant. The bigger the country, the more powerful its tyrant. (Barwise)

2. Every pie has a price. The smaller the pie, the lower the price. (Chris Fox)

3. Every man owns a donkey. He beats it. He feeds it rarely. (Fine)

4. Every child received a present. Jim opened it (his) immediately.

5. For every merchadise there is a price. For meet it is 10 euros.

6. Every child received a present. Some child opened it (his) immediately.

- In (1)-(6) the first sentence introduces a (functional) correlation

Henkin prefixes in natural language

1. Every country has a tyrant. The bigger the country, the more powerful its tyrant. (Barwise)

2. Every pie has a price. The smaller the pie, the lower the price. (Chris Fox)

- The second sentence states something about the correlation itself.

- Their expression requires a Henkin prefix.

Hintikka's subgame interpretation

5. Every child received a present. Jim opened it (his) immediately.

6. For every merchadise there is a price. For meet it is 10 euros.

7. Every child received a present. Some child opened it (his) immediately.

- The second sentence states something about a particular object in the range of the universal quantifier.

- Hintikka, subgame interpretation.

Functional dependence in natural language

3. Every man owns a donkey. He beats it. He feeds it rarely. (Fine)

4. Every professor at the university of San Clement_1 teaches a large lecture class_2. The professor_1 does all the grading of the class_2. The class_2 has a final exam_3. The final_3 is comprehensive. It_3 need not be long, however....

- The second sentence states something about the correlated objects.

Fine (1983, 1985): Arbitrary Objects

- A distinct domain of arbitrary objects
- An arbitrary object has instances which are individual objects
- Division: Independent (unrestricted) arbitrary objects and dependent ones

Dependence of arbitrary objects

- When *b* is an arbitrary object that depends only upon the arbitrary objects $a_1, a_2, ...$,then the values assigned to b must be determined upon the values assigned to $a_1, a_2, ...$, Example

- Let a be an arbitrary real

- Consider the arbitrary real a^3 which depends on a.The values assigned to a^3 depend on the values assigned to a :

a	a^3
1	1
2	8
3	27

- Consider the arbitrary real $\sqrt[3]{a}$ which depends on a. The values assigned to $\sqrt[3]{a}$ depend on the values assigned to a :

$\sqrt[3]{a}$	a
1	1
2	8
3	27

- The relation of value dependence

a	a^3
$\sqrt[3]{a}$	a
1	1
2	8
3	27

must be sustained by a relation of object dependence

- At the level of values: both a^3 depends on a and a depends on $\sqrt[3]{a}$.

- The table supports both of them.

The role of quantifier phrares (Fine)

- Quantifier phrases have two roles
- They make general assertions
- They introduce arbitrary objects
- Very roughly, universal quantifier phrases introduce unrestricted arbitrary objects
- Indefinites introduce dependent arbitrary objects
- The role played by scope is now performed by the relation of dependence between arbitrary objects.

Truth-conditions

- $\varphi(a_1, ..., a_n)$ is true iff it is true for all admissible assignments of individuals $i_1, ..., i_n$ to the objects $a_1, ..., a_n$.

- The set of admissible assignments: relation of dependence among arbitrary objects.

Identity criteria for arbitrary objects

- Suppose first that a and b are independent objects. Then $a = b$ iff their ranges are the same.

- Suppose now that a and b are dependent objects. Then $a = b$ iff:

1. a) They depend upon the same arbitrary objects; and b) their dependence ranges are the same.

2. They depend upon these objects in the same way.

Arbitrary objects in natural language (Fine)

> Every farmer owns a donkey. He beats it. He feeds it rarely...

...for there is no individual farmer or individual donkey to which the pronouns can sensibly be taken to refer. But countenance arbitrary objects and the difficulty disappears. 'He' refers to the arbitrary farmer, 'it' to the arbitrary donkey that he owns. Note this this arbitrary donkey is a dependent object and that for a given farmer as value for the arbitrary farmer, the arbitrary donkey can only take as a value a donkey that the farmer owns. Thus the statement 'He beats it' will be true, just as it should be, iff for all values i and j simultaneously assumed by the arbitrary farmer and donkey, it is true that i beats j. (Fine, 1983)

Objections against Skolem functions (Fine, Fox)

- Skolem functions cannot handle the cases involving multidependencies: c depends on b in *a particular way*, say $c = 2b$, and b depends on a in *another way*, say $b = a^2$:

a	b	c
a	a^2	$2a^2$
1	1	2
2	4	8
3	9	18
$\vdots$	$\vdots$	$\vdots$

- We can express this in IF logic by

$$\forall x \exists y (\exists z / \{x\})(z = 2y \wedge y = x^2)$$

which involves a signaling sequence.

- Their Skolemized version:

$$\forall x \left[f(g(x)) = 2g(x) \wedge g(x) = x^2 \right]$$

Signaling sequences: arbitrary objects

1. Every professor at the university of San Clement$_1$ teaches a large lecture class$_2$. The professor$_1$ does all the grading of the class$_2$. The class$_2$ has a final exam$_3$. The final$_3$ is comprehensive. It$_3$ need not be long, however....

- Arbitrary objects representation:

  $P(a) \wedge C(b) \wedge T(a,b) \wedge Gr(a,b) \wedge E(c) \wedge H(b,c) \ldots$

where b depends on a, c depends on b,...

Advantages of using arbitrary objects

- Binding scope has disappeared

- Syntactical scope has been replaced by the dependence relations between arbitrary objects

- However, at the level of truth-conditions, we need also individual objects

Representation in IF logic

- In the sentence

  Every marchandise has a price

- The co-variation takes place at the level of values and not at the level of "arbitrary objects".

- The covariation is indicated by the syntax

- Representation in IF logic:

$$\forall x(P(x) \to \exists y(C(y) \wedge T(x,y) \wedge Gr(x,y) \wedge (\exists z/\{x\})(E(z) \wedge H(y,z) \wedge ...)))$$

- The co-variation at the level of values requires sets of assignments

Brasoveanu & Farkas (2011)

1. Every x student in my class read a y paper about scope

2. $\forall x[\mathit{student}(x)]\ \exists y[\mathit{paper}(y)]\ \mathit{read}(x, y)$

3. $\forall y[\mathit{paper}(y)]\forall x[\mathit{student}(x)]\ \mathit{read}(x, y)$.

Brasoveanu & Farkas (2011): The *Binder Roof Constraint*:

> An indefinite cannot outscope a quantifier that binds into its restrictor

1. Every^x student read every^y paper that one^z of its$_y$ authors recommended.

2. Every^x student read every^y paper that a^z professor recommended

- In (1), "one$_z$ of its$_y$ authors" can have only narrowest syntactical scope.

- In (2), "a^z professor" can have only the narrowest syntactical scope

Brasoveanu and Farkas (2011): semantical scope (dependence)

- In (2) we can have 3 different forms of dependence:

(NS) For every student x, for every paper y such that there is a professor z that recommended y, x read y

$$\forall x \forall y \exists z Q(x, y, z)$$

(IS) For every student x, there is a professor z such that, for every paper y that z recommended, x read y.

$$\forall x \forall y (\exists z / \{y\}) Q(x, y, z)$$

(WS): There is a professor z such that, for every student x, for every paper y that z recommended, x read y.

$$\forall x \forall y (\exists z / \{x, y\}) Q(x, y, z)$$