On Multitudes

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Mostly harmless.

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1 Why Multitudes?

Metaphysics, according to the fine American philosopher Donald C. Williams, consists of two parts: *analytic ontology*, which comprises a detailed and painstaking analysis of the basic and most important entities, and *speculative cosmology*, which makes very general, bold and largely unverifiable conjectures about the nature of the world as a whole. My bold speculative conjecture is *naturalism*, according to which everything that exists is within and part of the spatial–temporal–causal world in which we find ourselves, which we perceive, and which we investigate in the natural sciences. Nothing else exists: no abstract objects outwith space, time and the causal nexus, no non-spatial minds or spirits, whether finite or infinite. This may seem like common sense but has always been a minority opinion amoung philosophers, who have accepted abstract entities like forms, universals and mathematical objects, non-spatial minds or spirits, both in ourselves and others, especially God. To deny these is to be a naturalist and nominalist, mind–body monist and atheist: I am all these.

One of the biggest problems facing a naturalist is to account for mathematics, a vast, ancient and noble discipline with millions of practitioners and thousands of books full of results, many of them of great subtlety and beauty. It is rather natural for mathematicians and others to be Platonists about mathematical objects, to consider them abstract, eternal, timeless and necessarily existent things that mathematicians find out truths about, somewhat as astronomers find out truths about heavenly objects. But Platonism in mathematics faces a problem endemic to all non-naturalist metaphysics, that of explaining how we can possibly have knowledge about such entities. Without arguing it here, I consider this problem to be insurmountable and so reject Platonism. So how does a naturalist account for mathematics? One plausible story is formalism, a view most closely associated with David Hilbert but pre- and postdating him. According to this view mathematical statements are not *about* anything, but get their validity from the logical connections among different statements of extremely general or formal character. The key notion in this explanation is the idea of one statement's following analytically from another, that is, following logically, given the meanings of the terms involved.

In early formulations of formalism by Hilbert and others, *following from* was defined in terms of the existence of a finite proof of one statement from another. But this view was overturned by the incompleteness results of Gödel, which showed that only some of the valid consequences of a statement could be formally proved. Rather the notion of valid consequence needs to be formulated differently, according to ideas first propounded by Bolzano in 1837 and rediscovered independently by Tarski in 1936. According to this, a statement *q* follows from a statement *p* if any model of *p* is a model of *q*. This means we must say what can serve as a model. While concrete individuals can compose models, in many circumstances such models are not guaranteed to exist, because we do not know whether there are only finitely many individuals in the world, but we need to allow for the possibility of infinite models of many statements. So model theorists take mathematical objects as models, where typically they select these models from mathematical theories, the most frequently used being set theory with some additional structure defined on the sets.

But now here is the rub: a naturalistic formalist cannot avail of sets and other mathematical objects as models, because they are abstract and so according to their view do not exist. To do so would be to cheat, or to have one's semantics be out of tune with one's ontology. Obviously such worries do not concern a Platonist, but logicians with a naturalist tendency face this problem and have adopted different stances to it. Quine, who have preferred to be a nominalist, admitted sets into his ontology in order to give mathematics, which he considered indispensable to science, a home. Field tries to undercut this by looking for models of successful scientific theories among physical things, but he has to treat necessity or consequence as a primitive, undefined notion, which seems wrong. Also Field's approach is hostage to the potential shortage of concrete objects for his models, and he can offer no similar account of pure mathematics without application. Leśniewski responded by rejecting any semantics that employed sets or other abstract objects, and pursuing his logical theories at the intuitive level, which is rigorous and perhaps noble, but unsatisfactory. His student Tarski used sets and other abstract objects for their semantic value, but remained privately sceptical about them. Seeing no way to do logic nominalistically, he described himself as a "tortured nominalist" and suffered a split intellectual personality. Boolos, like Leśniewski, employed plural terms to give nominalistically acceptable models for monadic predicate logic, of first and second order, but underestimated the difficult of accounting nominalistically for relations, which typically employ ordered pairs and other tuples, or else have to introduce other sorts of abstract entities such as relational universals.

Having endured this difficulty for years, because of accepting the position of Leśniewski, I now believe I see a way out, which is to recognize the existence of what I call *multitudes of higher order*. Unlike Leśniewski, who accepts and has terms for only multitudes of first order, this allows us to provide models in a way similar to that of set theory. But unlike sets, multitudes, of whatever order, are concrete if their members are concrete, and are necessarily such that if their members exist, then they exist. So multitudes are nominalistically acceptable, and if all their urelements (ultimate individual members) are natural, so are they. This closes the gap in the naturalistic account of mathematics and allows a naturalist to be a formalist with a clear ontological conscience.

2 A Biblical Example

I will introduce the idea of multitudes, of first and higher order, by means of examples. Obviously a formal or logical theory of multitudes is desirable, but this is not the place to go into such a theory. It will be sufficient for present purposes if the reader gets the idea of a multitude, whether or not she eventually accepts multitudes. Let me start with an example from the Bible: the story of Noah. Though the example is almost certainly fictional, it serves its purpose of introducing the ideas. Those sensitive to the possibly fictional status of the characters involved are welcome to substitute their own non-fictional example.

Having built the ark to avoid destruction in the flood, as God told him, the eight people selected to be saved from the flood enter the ark, as it says in the Book of *Genesis*, Chapter 7, Verse 7:

> And Noah went in, and his sons, and his wife, and his sons' wives with him, into the ark, because of the waters of the flood
>
> *et ingressus est Noe et filii eius uxor eius et uxores filiorum eius cum eo in arcam propter aquas diluvii*
>
> Noe wszedł z synami, z żoną i z żonami swych synów do arki, aby schronić się przed wodami potopu.

Consider the human population of the ark according to this story: they are eight people: Noah, his here unnamed wife, his three named sons Shem, Ham and Japheth, and their three unnamed wives. As I understand it, a multitude is simply several things, a plurality of things. In this case, the human occupants of the ark are a multitude of eight people. Other words may be used in place of 'multitude': 'plurality' or 'group' are possible choices. I will explain my choice of the term 'multitude' presently. For the moment I will simply attempt to convey by means of examples and informal explanations what I mean. Here are several more examples of multitudes:

the people in this room now

(= the people in the Senate Hall of the University of Opole at 12 noon on 18 October 2011)
the legs of these people now in this room
the Kings and Queens of Poland from 960 to 1795
the administrative regions (województwa) of Poland since 1999
Russell and Whitehead (= the authors of *Principia Mathematica*)
SS. Matthew, Mark, Luke and John (= the four evangelists)
the valid figures of categorial syllogistic according to Galen
the prime numbers
the odd prime numbers
the even prime numbers
the people in this room now, however many there are of them, and the first few prime numbers, one for each person in this room. (For example, if there are 82 people in this room, these will be the first 82 prime numbers.)

Each individual that is one of the several things mentioned in each case we may call a *member* of the relevant multitude. So I am a member of, am one of, the people in the room, as is Professor Wybraniec-Skardowska; Władysław II Jagiełło is one of the kings and queens of Poland, and St. Luke is one of the evangelists. I have included among the examples cases where the objects are abstract, such as numbers and syllogistic figures, not because I believe such things exist, but in order to show that one can be a Platonist and still accept multitudes of abstract objects, and indeed multitudes of mixed abstract and concrete objects, as in the last example. Anything that exists can be and inevitably is a member of at least one multitude.

Several individuals which are among the members of a certain multitude may be called a *submultitude* of the larger multitude: for example the Piast monarchs are a submultitude of the Kings and Queens of Poland. In general we can mimic many of the concepts of set theory among multitudes.

3 Multitudes, Collections, Sums and Sets

Multitudes are what plural expressions designate, but we have to be careful to distinguish them from other things amongs what may be generically termed collective entities or collections. Not all collections are multitudes, and indeed none of the examples I have given above is in all ways of understanding it simply a mere multitude. I will illustrate the distinction by considering an orchestra. As a concrete example I will take the Hallé Orchestra in Manchester, founded in 1858 by Karl Halle (Sir Charles Hallé) and the world's fourth oldest orchestra. Since its debut, the orchestra has had many hundreds of musicians and twelve principal conductors. At any one time, certain people are members of the orchestra. All of these were at one time not members, and most of them will either move on elsewhere or retire from the orchestra. The orchestra is made up of different musicians at different times, survives the coming and going of its members, and over time completely changes its membership. Take the members of the orchestra at a certain time, for example 1 January 2000. They are a multitude of people, all musicians. But this multitude, unlike the orchestra, does not change its membership, and by its nature as a multitude, it cannot. It is essentially just these individuals. Unlike the membership of an orchestra, which is indexed to times, membership of a multitude is not something an object can have at one time and not another: and being a member or one of these several things, is not indexed by time at all. So although this multitude at one time constituted the orchestra, at other times it did not.

Although the members of a multitude may exist in time, they need not all exist together. For example the Kings and Queens of Poland did not all exist together at any time. And as some of the examples indicate, if there are individuals such as numbers or syllogistic figures that are not temporal at all, there can still be multitudes of them. The single important fact about a multitude is what its members are. The existence and identity conditions of a multitudes turn exclusively on its members. A multitude exists only if it has at least one member that exists, and the identity of a multitude is determined solely and completely by what members it has: multitudes with the same members are identical. Multitudes are purely extensional collections.

Multitudes are similar in these ways to sets. The elements of a set are not elements at a time, they simply are elements. And sets with the same elements are identical. There are however two crucial differences between sets and multitudes. Firstly, while a multitude of concrete individuals, say the people in this room, is concrete, and currently occupies a volume which is the sum of the volumes occupied by the individual members, any set is abstract and timeless, even if its elements are not. The set of people in this room is not in this room, ever, whereas the multitude now is, because all the people now are. A multitude is nothing over and above its several members: it is them and they are it, whereas a set is something extra, a new, additional individual. It is therefore theoretically possible to accept that all the people in this room exist but deny that their set exists. That is my own position. On the other hand it is inconsistent to accept that each of the saints Matthew, Mark, Luke and John exists but deny that the four of them exist. Therefore once we accept that several individuals exist, we are forced to accept that the multitude of them exists: the multitude is an automatic concomitant whereas the set is not. The second important difference between sets and multitudes is that there is a null or empty set, the one having no elements, but no empty multitude. There can be nothing which is no things. This again proves that sets are additional entities, because the empty set exists even though there is nothing that is its element. I consider that much of the ontological appeal and even supposed harmlessness of sets arises from confounding them with multitudes.

Another category of entities which can easily be confused with multitudes are mereological sums or fusions. Like multitudes, these are concrete if their components are, and there is no empty or null sum. But again there are crucial differences. A mereological sum of several things is one individual, whereas the several things are not one, but precisely several individuals. It is of multitudes, but not sums, that we can truly say they are two, three, four etc. things, i.e. we can predicate numbers other than *one* of them. This was recognized over a century ago by Bertrand Russell, who called multitudes 'classes as many', and contrasted them with 'classes as one', which in Russell's unclear usage are a little like sets but also a little like sums. As in the case of sets, there is no contradiction in denying that the mereological sum of two or more individuals has to exist if the individuals exist; indeed quite a few mereologists do, myself included. Only those who think that every multitude yields a sum are likely to make this mistake.

Having said what multitudes are and what they are not, let me justify the choice of name, again by reference to the Bible. In St. Luke's version of the Christmas story, when the angel appears to the shepherds, he is accompanied by many other angels forming a heavenly army or host. In the King James English translation the start of verse Luke 2:13 comes out as:

> And suddenly there was with the angel a *multitude* of the heavenly host praising God and saying …

while in the Latin and Polish versions respectively we have:

> *et subito facta est cum angelo* multitudo *militiae caelestis laudantium Deum et dicentium…*

> I nagle przyłączyło się do anioła *mnóstwo* zastępów niebieskich, które wielbiły Boga słowami…

As standardly used of course, words like 'multitude' imply a large number. But we will stretch the meaning of the term to allow a pair of things to be a multitude, and even a single individual. An individual as a multitude is nothing new: it is just the individual itself. Also multitudes standardly so called consist of lots of things close together or taken together in some way, but we will not insist on this: according to our usage a multitude can be scattered and heterogeneous: there are absolutely no constraints on what its members can be.

I like the term ‘multitude’ because it is in its everyday meaning close to what is intended, and is not otherwise used in philosophical ontology. In the past I have used the terms ‘manifold’, ‘class’ and ‘plurality’, and in German *Vielheit*. No term is ideal. Even ‘multitude’, its biblical heritage nothwithstanding, has connotations of large size and intermember proximity, and it is inconveniently trisyllabic. A nice monosyllabic word is ‘group’, and ‘group of groups’ is trisyllabic whereas ‘multitude of multitudes’ is heptasyllabic. I am tempted to use ‘group’: only its widespread use and prominence in mathematics prevents me from making the switch.

In sum then, our basic idea is that anything that can be counted can be reckoned as a member of a multitude. Since multitudes themselves can be counted, this idea will have far-reaching consequences.

4 The Neglect of Multitudes

Collective entities in general, and multitudes in particular, have been unjustly ignored or assimilated to other things for a long time. They are ontological Cinderellas. The person chiefly responsible for this is probably Gottlob Frege, because in his logic Frege understood all names or terms to be singular, both syntactically, and for scientific purposes, semantically. That is, when a nominal expression functions properly, it names a single individual, and it only goes wrong when it seems to name an individual but does not. By contrast in traditional logic from Aristotle to the twentieth century, a name or term could name several individuals, or indeed none at all. Thus whereas ‘Socrates’ names the famous Athenian philosopher, ‘philosopher’ names many people. For Frege, ‘philosopher’ is not a term, but a morphological fragment of a predicate, ‘… is a philosopher’, which names no one but is true of several people. Frege’s practice was taken up by modern logic through Russell and others, and is now logical orthodoxy.

Now Frege was right that there are important differences between a name like ‘Socrates’ and a common noun like ‘philosopher’. But that does not mean that common nouns do not denote or stand for the several individuals that are philosophers. Further, there are plural referring expressions like ‘the authors of *Principia Mathematica*’ or ‘Whitehead and Russell’ which function just like singular terms in that they refer to certain specified individuals. The only difference between such names and others such as ‘the author of *The Critique of Pure Reason*’ and ‘Immanuel Kant’ is that they are plural rather than singular, both syntactically, in that they carry singular inflexions, and semantically, in that they stand for more than one individual.

A number of logicians have however refused to follow Frege and the rest in restricting names or terms to the singular. Foremost among them is Stanisław Leśniewski, who in his logical system called ‘ontology’ allows names to be empty or plural as well as singular. More recently the logician George

Boolos introduced plural terms into his logic. My own first foray into this terrain occurred in a short 1980 article entitled ‘Individuals, Groups and Manifolds’, and continued at greater length in two essays of 1982 called respectively ‘Number and Manifolds’ and ‘Plural Reference and Set Theory’. The term ‘manifold’ was the one I then used for multitudes. The inspiration for my articles came from articles by Eric Stenius and Max Black, who had also protested against the Fregean orthodoxy. It soon became apparent to me that Leśniewski’s ontology already embodied as sound a logical system for such plural terms as one could envisage, so there was no point in continuing with alternatives as I had been doing. That was and remained my position until very recently.

5 First-Order Multitudes

The most obvious examples of multitudes are multitudes whose members are all individuals, as in the examples we have given so far. Provided we confine our attention to such multitudes, there is no beating the elementary part of Leśniewski’s ontology, what Jerzy Słupecki called Leśniewski’s ‘calculus of names’, but which could equally well have been called ‘calculus of multitudes’. In the best-known formalization of this system, using as primitive the functor written ‘ε’ and meaning ‘is one of’, the axiom is

AxOnt $$a \,\varepsilon\, b \leftrightarrow \exists c \ulcorner c \,\varepsilon\, a \urcorner \wedge \forall c \ulcorner c \,\varepsilon\, a \rightarrow c \,\varepsilon\, b \urcorner \wedge \forall cd \ulcorner c \,\varepsilon\, a \wedge d \,\varepsilon\, a \rightarrow c \,\varepsilon\, d \urcorner$$

and the scheme for admissible nominative definitions is

NomDef $$a \,\varepsilon\, \Phi(b\ldots) \leftrightarrow a \,\varepsilon\, a \wedge \varphi(a,b,\ldots)$$

where the definitions conforming to this scheme have to fulfil a number of conditions aiming to avoid circularity, paradox and other pathologies of definition. In particular the first clause on the right-hand side ‘$a \,\varepsilon\, a$’ is there to prevent defining a multitude of non-self-membered multitudes in the fashion of Russell’s Paradox, the subject of paradoxes being one which obsessed Leśniewski for many years. In this system we may define an indefinite number of new predicates and nominal functors. An indication of what is possible in this way is given in Leśniewski’s 1929–30 lecture notes published in 1988.

It is worth noting a number of things about Leśniewski’s system for future reference. The first is that because of Leśniewski’s non-commital way of understanding quantifiers, to state a proposition quantifying a variable does not itself commit the logician to the existence of anything, individual, multitude or any other entity. It is a point of pride for Leśniewski that his logical theorems are true for all possible circumstances, including the limiting case in which nothing at all exists. This is a feature we wish to preserve in what follows. We shall also continue to understand the quantifiers in the non-commital way. Two of the most important logical constants that Leśniewski is able to define in his ontology are a universal name and an empty name. Their respective definitions go as follows:

Def. V $\quad a \, \varepsilon \, \mathrm{V} \leftrightarrow a \, \varepsilon \, a$

Def. Λ $\quad a \, \varepsilon \, \Lambda \leftrightarrow \mathrm{a} \, \varepsilon \, a \wedge \sim(a \, \varepsilon \, a)$

Both of these definitions conform to Leśniewski's scheme and conditions. The former is the special case where there is no second conjunct to the definiens. We may read 'V' as 'object', 'entity', 'thing' or 'individual'. In this system it is the broadest name possible and designates all individuals. The other name is deliberately designed to have an explicitly contradictory definiens and as a result can never designate anything. It is a theorem of ontology that $\sim\exists a \ulcorner a \, \varepsilon \, \Lambda \urcorner$. Because we can with some care read 'Λ' as 'nothing' we can read this theorem as 'nothing is nothing'. I like to call this 'Heidegger's Law'. While it is certainly not what Heidegger meant by *Das Nichts nichtet* it shows, contrary to Carnap, that this supposedly nonsensical sentence can be given an interpretation according to which it is not only meaningful but logically true! It is however not a theorem that $\exists a \ulcorner a \, \varepsilon \, \mathrm{V} \urcorner$, for this means that some individual exists. While this is true, it is not logically true, since it does not pertain to the case where nothing at all exists. In that exceptional circumstance, and in that only, the names 'Λ' and 'V' would be coextensive, since neither would designate anything. In all other circumstances, 'V' designates at least one individual.

Two names are coextensive when they designate the same individuals. Leśniewski subscribes to a strong extensionalism according to which all names, predicates and other functors conform to laws of extensionality. While this is in general a disputable position, for dealing with multitudes it is not controversial so we shall conform to this restriction. The upshot is that identity as applied to names satisfies Leibniz's Law:

Leib $\quad a = b \leftrightarrow \forall \varphi \ulcorner \varphi a \leftrightarrow \varphi b \urcorner$

In Leśniewski this is a theorem following from his definition of identity

Def.= $\quad a = b \leftrightarrow \forall c \ulcorner c \, \varepsilon \, a \leftrightarrow c \, \varepsilon \, b \urcorner$

and a general principle of extensionality.[1] In the more liberal environment we shall consider below it is preferable to treat Leib as the *definition* of identity.

It is a consequence of Leśniewski's way of treating multitudes that the number of individuals there are directly determines how many multitudes there are, such that if there finitely many individuals, say *n*, there are finitely many multitudes, namely $2^n–1$. The missing case is that of Λ,

[1] Leśniewski uses the symbol '=' for singular identity, but we are not here following his usage.

which is not a multitude and does not exist. Nevertheless if we define a submultitude in the obvious way

Def.⊂ $a \subset b \leftrightarrow \forall c \ulcorner c \,\varepsilon\, a \rightarrow c \,\varepsilon\, b \urcorner$

then it follows that for any a, $\Lambda \subset a$. So if a has n members (we write $||a|| = n$) then the number of extensionally distinct names we can substitute for 'b' such that '$b \subset a$' is true is indeed 2^n. In this slightly naughty sense we can say that a has 2^n submultitudes.

6 Multitudes of Multitudes

There are at least four reasons why we should accept the existence of multitudes. The first, and relatively weakest, is that we do have expressions for them: plurals and other collective expressions. But not all language should be taken at face value. Secondly, we can distinguish one multitude from another, by their members. Thirdly, as a result of this there are truths about multitudes which are not truths about their members. For example it is true of the evangelists, as it is of the Beatles, the seasons, and the constituent nations of the United Kingdom, that they are four. Each of the members is not four but one. Only the multitude is four. There are many things which are true of multitudes that cannot be true of individuals. For example a multitude of soldiers may surround a town, or win a battle, or lift a heavy object such as a car or a tree.

There are then solid theoretical reasons to distinguish an individual from a multitude, and one multitude from another because one has an individual as a member that the other does not. We will argue that these reasons likewise force us to conclude that there are not only multitudes of individuals but also multitudes of multitudes. Take our first example from the bible: Noah; his sons; his wife; his sons' wives. The biblical text divides this group into these four subgroups in all occasions when it is mentioned, and they are always mentioned in the same order (men before women). This gives us four multitudes, counting the singleton multitudes Noah and Noah's wife as two of these. Calling the wives of Noah, Shem, Ham and Japheth by their names not from the Bible but from the *Book of Jubilees*, respectively, Emzara, Sedeqetelebab, Ne'elatama'uk and 'Adataneses, then we have four multitudes

Noah (N)

Shem, Ham and Japheth (S, H, J)

Emzara (E)

Sedeqetelebab, Ne'elatama'uk and 'Adataneses (Q, K, A)

Let us notate this by separating the multitudes by vertical lines. This gives

N | S H J | E | Q K A

On the other hand if we group the eight survivors of the Flood into their marital pairs, we have the following four multitudes:

N E | S Q | H K | J A

while if we group them according to generations we have

N E | S H J Q K A

and finally if we group them by gender we get

N S H J | E Q K A.

I claim that each of these four groupings is different, and that the multitude of eight survivors can be subdivided in different ways. So we should distinguish the four multitudes N E | S Q | H K | J A from the two multitudes N S H J | E Q K A, even though they are all built out of and based on the same individuals. The reasons we gave before for distinguishing multitudes from individuals continue to apply here: there are different (formal and material) properties that characterize them. One is four multitudes of two, the other is two multitudes of four. The multitudes of two are married couples; the multitudes of four all each of the same gender.

We thus have as good reason to distinguish multitude of multitudes of individuals from multitudes of individuals as we have for distinguishing multitudes of individuals from these individuals. In other words, we should recognize multitudes of multitudes, or second-order multitudes. But once we allow this, there is no theoretical reason to stop there. We can distinguish third-order multitudes, and so on from there.

There are precedents among higher-order collectives which are not multitudes. A football club such as Manchester City has several members, individual players like Arsevin and Fabregas. Manchester City is itself a member of another organisation, the (English) Football Association, whose members are clubs, not players. The FA, along with 207 other such national football organisations, is a member of FIFA, the Fédération Internationale des Football Association. Even FIFA might belong to a yet higher order of collective, consisting of international sports federations like the ICC (International Cricket Council), the IAAF (International Association of Athletics Federations) and others. We could imagine it called FIFIS, or something similar. And there could even be other international federations of international federations, for example for academic subjects (FISP is like FIFA for philosophy, for example). Only practicality and fantasy limit the orders. In the case of multitudes, such limitations are absent.

For reasons which are unknown to me, Leśniewski did not accept or recognize higher-order multitudes. And until recently, neither did I. As far as I can recall, my reasons were that there were no compelling reasons to accept them, and that to do so would be to open the door to considerable complications. But I now think there are compelling reasons, and that to avoid complications simply because they are inconvenient, or a lot of work, is simply laziness. If the world exhibits complications, it is up to us as scientists to recognize them and make our language as rich, and our theories and representations as complicated, as the phenomena warrant. This principle applies just as much in philosophy as in physics, biology or history.

7 Singletons and the Argument for Higher-Order Multitudes

One of the structurally notable principles of standard set theory is the distinction between a singleton set and its sole member. We have in general that $x \neq \{x\}$. The analogous principle does not hold for multitudes, because if *A* is an individual, it is its own singleton multitude. Using square brackets to group names in finite lists, we have that where *A* is an individual, *A* = [*A*].

Together with the existence of an empty set ∅, the principle that a singleton is different from its member allows an infinite collection of sets to be generated from the empty set, since ∅, {∅}, {{∅}}, {∅,{∅}} etc. are all distinct. This allows set theory to have models in which nothing is not a set, giving pure set theory, and these models are sufficient for accommodating many mathematical theories. Multitude theory does not have an empty multitude. In this it conforms with common sense, and with the ontological sensitivities of one of set theory's most trenchant critics, Leśniewski. There is no pure multitude theory. There can be no multitudes if there are no individuals. Hence multitude theory makes no ontological commitments, unlike standard set theory or standard logic. Unlike set theory, there is no magic, getting something out of nothing.

The standard argument, going back to Peano, for why set theory distinguishes *x* from $\{x\}$ is that x might itself be a set with more than one element. If that is so, *x* has several elements, whereas $\{x\}$ has only one, so by Leibniz's principle $x \neq \{x\}$. The same reasoning applies with multitudes. Suppose [A,B] is the pair multitude with two individual members A and B. We assume throughout that A ≠ B. Then [A,B] has two members. Now if we allow '[[A,B]]' as a well-formed expression at all, there are then only two possibilities: either [[A,B]] = [A,B], or [[A,B]] ≠ [A,B].

Let us look at the first option. It effectively says that, notationally, nested brackets can be reduced to a single pair, and ontologically that singletons are identical with their members, whether these are individuals or not. Consider then four individuals A, B, C and D. In our exposition of the Noah example earlier we assumed that different groupings of several individuals give rise to different second-order multitudes. So consider the two groupings [[A,B],[C,D]] versus [[A,C],[B,D]]. If [[A,B]] = [A,B] then adding another thing, say e, to [[A,B]] will give us [[A,B], e] and this will have the same effect as adding e to [A,B], so [[A,B],e] = [A,B,e]. Conversely, adding another item f to [[C,D]] to give [f,[C,D]] is the same as adding it to [C,D] to give [f,C,D]. If e = [C,D] and f = [A,B] this tells us that [e,f] = [[A,B],[C,D]] = [A,B,[C,D]] = [A,B,C,D]. But then if g = [A,C] and h = [B,D] then [g,h] = [A,B,C,D] also. Therefore [e,f] = [g,h] even though none of e, f, g and h are identical. This tells us two things. The first is that higher-order multitudes collapse down to first-order multitudes: there are no distinct higher-order ones. The second however is that it tells us that multitudes are not determined solely by their members, for then we would expect that [e,f] ≠ [g,h], since all four of the items are different.

I conclude from this that we should prefer the second option, which is that [[A,B]] ≠ [A,B], as this allows us to sustain the intuitively expected result that [[A,B],[C,D]] ≠ [[A,C],[B,D]]. This admits higher-order multitudes as distinct from first-order ones, and goes agains the "levelling" of multitudes to first order found in Leśniewski and in my own earlier writings. So there are three positions one may take about multitudes. One can reject them altogether, as in Frege; one can accept first-order multitudes only, as in Leśniewski; or one can accept higher-order multitudes as well, as I do.

There is a price to be paid for accepting higher-order multitudes, if one is a nominalist. It is that different entities may be "built out of" the same individuals, or based on the same urelements. In our example the four individuals A, B, C and D give rise to the second-order multitude [[A,B],[C,D]] and also to the distinct second-order multitude [[A,C],[B,D]], and to many others. This violates a nominalistic principle due to Nelson Goodman, that there shall be "No distinction of entities without distinction of content."[2] We have already violated Goodman's definition of nominalism, that it "consists in the refusal to countenance any entities other than individuals"[3] by recognizing first-order multitudes. Though there are possible interpretations of the former principle which outlaw first-order multitudes as well, it is well and truly violated by second- and higher-order multitudes. Therefore anyone who accepts high-order multitudes is, from a Goodmanian point of view, accepting what Roger Taylor's lyrics for the rock group Queen say: "It's a kind of magic." Accepting higher-order multitudes also brings conditional commitment to many new entities, indeed arguably infinitely many, even if there are as few as two individuals.

We argued for the distinctness of a multitude from its several members by pointing out that it has properties none of them have, in particular numerical properties like being four or being eight. The same reasoning applies to higher-order multitudes. Returning to our Noah example, the four multitudes

N | S H J | E | Q K A

N E | S Q | H K | J A

N E | S H J Q K A

N S H J | E Q K A

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[2] Goodman 1977, 26.

[3] Ibid.

or in the less perspicuous bracket notation

[N,[S,H,J],E,[Q,K,A]]

[[N,E],[S,Q],[H,K],[J,A]]

[[N,E],[S,H,J,Q,K,A]]

[[N,S,H,J],[E,Q,K,A]]

differ in several of their numerical properties: two have four members, two have just two, one of the two-membered ones has two four-membered members, the other has one pair and one six-membered member, and so on. So by parity of reasoning, once we accept first-order multitudes at all we should accept higher-order ones. These two options, viz.: no multitudes, or multitudes of any order, appear to be the only stable and reasonable ones. Leśniewski's middle position is inherently unstable.

8 Principles for Multitude Theory

So if we weaken the conditions on membership so that it is no longer transitive as it is in Leśniewski, what do we get? Without implying certainty or anything other than provisional status, here are the principles I think should govern multitudes of first and higher order.

Primitives

'=', meaning identity
'η', meaning 'is one of' or 'is a member of'

Axioms

Identity $a = b \leftrightarrow \forall\varphi\ulcorner \varphi a \leftrightarrow \varphi b\urcorner$

Extensionality $\forall c\ulcorner c\ \eta\ a \leftrightarrow c\ \eta\ b\urcorner \rightarrow a = b$

Existence $a\ \eta\ b \rightarrow \exists c\ulcorner c\ \eta\ a\urcorner$

Individuality $\forall c\ulcorner c\ \eta\ a \rightarrow a\ \eta\ c\urcorner \rightarrow (b\ \eta\ a \rightarrow b = a)$

Continuation $\forall h\ulcorner\forall a\ulcorner \mathrm{E}a \rightarrow \mathrm{E}ha\urcorner \rightarrow \forall a\ulcorner \mathrm{E}a \rightarrow \exists b\ulcorner a\ \eta\ b \wedge \forall c\ulcorner c\ \eta\ b \rightarrow hc\ \eta\ b\urcorner\urcorner\urcorner\urcorner$

Disjoint Choice $\forall ab\ulcorner a\ \eta\ c \wedge b\ \eta\ c \wedge a \neq b \rightarrow {\sim}\exists c\ulcorner \mathrm{c}\ \eta\ a \wedge c\ \eta\ \mathrm{b}\urcorner\urcorner \rightarrow \exists \mathrm{b}\ulcorner\forall \mathrm{a}\ulcorner a\ \eta\ b \leftrightarrow \exists d\ulcorner d\ \eta\ c \wedge a\ \eta\ d \wedge \forall e\ulcorner e\ \eta\ c \wedge a\ \eta\ e \rightarrow e = d\urcorner\urcorner\urcorner\urcorner$

Definition Frame

Nominal Definition $a\ \eta\ \Psi(b\ldots) \leftrightarrow \exists c\ulcorner c\ \eta\ a\urcorner \wedge \psi(a,b\ldots)$

I have simply baldly stated these and not justified them. The principal justification lies in the idea that any group of objects are a multitude, and that multitudes differ if, and only if, they have different members. These are probably not all the principles that are needed. In particular we would expect something like an Axiom of Foundation, to the effect that all distinctions among multitudes are traceable to distinctions among individuals. The reason no such principle is stated here is simply that I have not settled on how properly to formulate it. At any rate, this is the general area within which a formal theory of multitudes of arbitrary order could be grown. A major source of the expressive strength of the system here outlined is the principle of nominal definitions, which allows among other things a principle of comprehension to be derived, and power multitudes, singletons and pairs, and a universal and empty multitude to be defined, along lines exactly analogous to those of Leśniewski, substituting 'η' for 'ε'. On the other hand it can be shown that a Russell-style multitude must be empty. So at least one obvious paradox cannot arise. And nothing can arise from nothing, as it can, miraculously, in set theory. As given. the principles are true for a domain of no individuals or one individual. With two or more individuals, it can be shown that they imply that there are infinitely many multitudes. This is as we should expect: multiplicity arises from difference. If the principles are in fact inconsistent on a domain of more than one individual, that would be an interesting and important result: so far I have neither a consistency proof nor a proof of inconsistency.

9 Why Higher-Order Multitudes are Nominalistically Acceptable

I shall argue that there is no reason from a nominalistic point of view, Goodman aside, why one should want to resist the acceptance of multitudes of any order. Suppose we have three individuals A, B and C. Then in addition there are the four multitudes AB, BC, AC and ABC. I claim that you cannot accept the existence of each of the three individuals A, B and C but deny existence of the trio ABC. In this otmultitudes differ from sets and also from mereological sums. There is no contradiction in thinking that A, B and C exist but that the set {A,B,C} does not, if one does not think that sets exist. That was the position of Leśniewski, and it is also my own view. Likewise, those, also like myself, who deny that in mereology there is always a mereological sum A+B+C of any three individuals, consider that A, B and C may exist but the sum not. A multitude is as concrete as its members, so if A, B and C are all concrete individuals, the multitude ABC is also concrete, but of course not an individual. ABC has a location, which is the sum of the locations of A B and C: if A B and C have weights and volumes, and do not overlap, the weight of the multitude ABC is the sum of the weights of A B and C and the volume is the sum of their volumes.

Likewise the higher-order multitude AB | BC (or [[A,B],[B,C]]) has properties similar to those of its ultimate members. It is where its ultimate members are, and its properties derive from theirs. But because the two members AB and BC share a member, B, the weight or volume etc. of B only counts once. It is not the proliferation or multiplication of new properties that tells us we have new objects, but the fact that the new objects are not identical with any of the others. As a result, the world does not get any more crowded or heavier because there are multitudes, of first or higher order. To Frege, to Leśniewski, to myself, to everyone, it looks the same. Given the existence of individuals, the existence of multitudes of which these are members, and multitudes having these multitudes as members, and so on, follows, I claim, logically. But standard logic does not recognize this. This tells that what we need a new logic, in which multitudes feature as first-class named objects. The nearest equivalent to this in previous writing is as far as I know Quine's non-standard set theory NF from his 1937 'New Foundations for Mathematical Logic'. Like Quine, I think there is a universal multitude, the multitude of absolutely everything, and I think that individuals are their own singletons. But the logic I think is right for multitudes differs from Quine's set theory in three ways. Firstly, it is not about sets. Secondly, it has no null multitude. And thirdly, we lay no stress on stratification per se as Quine does. Otherwise I am very unclear about the relationship, despite their having some interesting features in common.

In saying that multitudes, of first and higher order, are nominalistically acceptable, I am saying two things. The first is that multitudes are not abstract objects in the way that Platonists and their like understand these. The second is that, provided we are prepared to accept any multitudes at all, even the pair whose members are this table and that table, then we are logically bound to accept not only all multitudes of individuals, but are also bound to accept higher-order multitudes, on pain of inconsistency. So although multitudes are, by this reasoning, something in addition to individuals, and not simply an illusion projected by language, they are an addition that comes for free, as part of the package, along with the individuals. In terms I do not especially like using, multitudes *supervene* on their urelements. Since supervenience is often seen as a "lite" form of ontological commitment, the extra commitment brought by accepting multitudes seems not to be ontologically problematic.

Nevertheless I would dispute that it is a "lite" or minor addition to individuals. Conditional upon the existence of a minimal number of individuals, it in fact implies the existence of an infinity of things. Given three individuals A, B and C, we can readily see that there are new multitudes of every finite order, and so infinitely many of them. In fact I think that two individuals suffice for there to be infinitely many multitudes: we have A, B, [A,B], [A,[A,B]], [B,[A,B]] and [A,B,[A,B]] in the lowest three orders. These are six entities, and since three entities suffice to generate an infinity, two individuals generate an infinity. As to whether there is a multitude with infinitely many members, even with finitely many urelements, while the principles I have mentioned to date do not imply either that there is or there is not, it seems to me that reasonable existence conditions imply that there is. These conditions in fact imply a very simply theorem of comprehension, so that all we need to do to be sure of the existence of an infinite multitude is to formulate a suitable condition making it clear that there is such a multitude, given the minimal antecedent condition that at least two urelements exist. Note however that fewer urelements, namely one or zero, do not suffice to generate an infinity of multitudes. In this respect, multitude theory contrasts starkly with standard and non-standard set theory, and much to its advantage.

Multitudes contrast with other kinds of collections in that they require nothing more for their existence than the existence of their members. By contrast a football team requires there to be (at least) eleven players who play together as a team. The existence of the eleven players is a necessary but not a sufficient condition for the existence of the team. By contrast the existence of the eleven members is both necessary *and sufficient* for the existence of their multitude.

10 Uses of Multitudes

I claim that once we accept the existence of multitudes we are required to reform our logic so that it can cope with them. Of course we can choose not to do so, and that is perfectly reasonable, but it is then a restriction, since multitudes, which I claim exist if at least two individuals do, are not being recognized and given their proper place. Developing a logic taking multitudes into account is a task for another occasion.

What benefit can multitudes bring to us? One benefit is that it gives us a natural ontology for cardinal arithmetic. Multitudes are the bearers of numerical predicates: Noah and his family have the property of being eight, the administrative voivodeships of Poland since 1.1.1999 have the property of being sixteen, and so on. We do not need to advance to a higher type to have the bearers of non-trivial numerical properties: multitudes will do. We even have what Frege and Russell would have called the number eight, that is, the multitude of all eight-member multitudes. But the number eight as an abstract entity is not to be found that way. Rather it is a product of abstraction under the equivalence relation of equinumerosity among all eight-membered multitudes. As such it is suspicious for a nominalist. While there are many eight-membered multitudes, and in our view also the multitude of all eight-membered multitudes (which is approximately what Frege and Russell would consider to be the number eight), for a nominalist the number 8 as an abstract individual does not exist.[4]

The most striking role that multitudes can perform is a logical one. Standard logical semantics uses sets. A nominalist cannot accept sets. Tarski's use of sets in his semantics is the reason why his

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[4] Cf. Simons 2007.

teacher Leśniewski did not accept Tarski's theory of truth for formal languages. Quine, who in many respects (apart from sets and quantification) is close to Leśniewski in his attitudes in logic and metalogic, claimed that while first-order logic does not commit us ontologically to sets, higher-order logic does, or else it commits us to entities such as properties and relations, or propositional functions, that Quine considers ontologically more dubious than sets. Famously, Quine described higher-order logic as "set theory in sheep's clothing",[5] in other words, a proper interpretation (and not just the semantics) requires sets but the language of higher-order logic disguises this.

It has been generally accepted since Gödel's incompleteness results that the proper account of logic's prime concept, namely logical consequence, is semantic rather than proof-theoretic. A sentence *q* follows logically from a sentence *p* if and only if every model of *p* is a model of *q*. With multitudes at our disposal, we can provide models for sentences which do not require the use of any abstract objects. It is thus possible to give a semantics for first- and higher-order logic that a nominalist can accept, contrary to Leśniewski who thought that such a semantics was not desirable, and contrary to Quine who thought such a semantics could not be nominalistic. The details of how to construct such models involve a certain amount of artificiality, but that is not a problem because models are not required or intended to represent the linguistic meaning of the sentences modelled: they are precisely models, and perform a formal semantic role. All that matters for their role as models is that they have a suitable structure: higher-order multitudes offer this possibility in a nominalistically acceptable way.

11 Conclusion

I hope to have introduced multitudes in a way which makes them seem quite natural and, from an ontological point of view, if not completelz anodyne, then mostly harmless. The task of developing an adequate logic for them is a difficult and more daunting matter, and requires some deep revisions in accustomed ways of thought. I have only stated some principles. Finally I hope it is clear from the last section that there are considerable theoretical advantages to recognizing the existence of multitudes.

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[5] Quine 1986, 64.

References

Goodman, Nelson. *The Structure of Appearance*. 3rd ed. Boston: Reidel, 1977.

Quine, W. V. *Philosophy of Logic*. Cambridge, MA: Harvard University Press. 2nd ed. 1986.

Simons, Peter. Individuals, Groups and Manifolds. In R. Haller and W. Grassl, eds. *Logic, Language and Philosophy*. Vienna: Hölder-Pichler-Tempsky, 1980, 483–6.

—— Number and Manifolds. In B. Smith, ed. *Parts and Moments*. Munich: Philosophia, 1982, 160–98.

—— Plural Reference and Set Theory. Ibid., 199–256.

—— What Numbers Really Are. In R. E. Auxier and L. E. Hahn, eds., *The Philosophy of Michael Dummett*. La Salle: Open Court, 2007, 229–247.